COMMENTARII MATHEMATICI
UNIVERSITATIS SANCTI PAULI
Vol. 36, No. 2, 1987

ed. RIKKYO UNIV/MATH
IKEBUKURO TOKYO
171 JAPAN

Submodules of Direct Products

by

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(Received July 9, 1986)

Abstract

In recent years the additive group of power series with integer coefficients has received the attention of several authors. Applications to analytic function theory may be found in [4]. In this paper we study the ring of power series with integer coefficients and coordinate-wise operations. As a special case of more general results we give a complete determination of the structure of the pure ideals (submodules) of this ring and a complete determination of the structure of those ideals which are direct summands.

§ 1. The concepts and definitions

Let Π denote the direct product of euclidean domains R_i, where i ranges over an indexing set N. The elements of Π are added and multiplied coordinate-wise. Π may be treated as a module over itself under the above coordinatewise addition and multiplication.

If x is an element of Π then x is a function from N into the union of the R_i's with the property that $x(i) \in R_i$. After Fuchs, see [2] § 97, p. 172, we define the support of x to be set supp X given by $\{i \in N \mid x(i)$ is not the zero element of $R_i\}$. If X is a subset of N we let h_X denote the characteristic function of X. Then h_X is the element of Π whose i-th coordinte is the zero element 0_i of R_i if $i \notin X$ and whose i-th coordinate is the identity element 1_i of R_i. Thus I_N (or simply I) is the identity element of the ring Π.

Although we are primarily interested in the case in which the cardinality $|N|$ of N is at most countable, such restrictions when required will be made explicitly.

DEFINITION 1.1. A subset W of Π is said to be *prim* if and only if $y \in W$ implies that $h_{\mathrm{supp}\, y} \in W$.

Note that a prim subgroup (Z-submodule, Z being the ring of integers) of Π is generated by characteristic elements, and hence is a Specker group, see [2], § 97.

DEFINITION 1.2. Let M be a module over a ring R, and let W be a subset of M. We shall say that W is R-pure (or just pure where there is no confusion) if whenever an equation $xr = w$ with $r \in R$, $w \in W$ has a solution $x \in M$, the equation also has a solution y in W.

Remark. A subset W can be prim without being pure or can be pure without being prim. Let Z denote the ring of integers. Then this remark is valid even for Z-submodules M of Π and not just for subsets of Π. To see that primness does not imply purity, let N consist of one element i and let R_i be the ring Q of rational numbers. Then $\Pi = Q$. Let M denote Z. Then Z is prim in Π but is not Z-pure. Next we show that purity does not imply primness. Let N consist of the two elements 1 and 2. Let $R_1 = R_2 = Z$. Let M be the Z-submodule of $\Pi = Z \oplus Z$ generated by (1, 2). Then clearly M is Z-pure but not prim.

Notation 1.3. Let Σ denote the additive subgroup of Π generated (as a group) by the characteristic elements of Π which have finite supports. Let B denote the additive subgroup of Π generated (as a group) by all the characteristic elements of Π. Let Σ_Π and B_Π be the Π-submodules of Π generated by Σ and by B respectively. Then clearly $B_\Pi = \Pi B = \{xb \mid x \in \Pi \text{ and } b \in B\}$ and so $B_\Pi = \Pi$ and $\Sigma_\Pi = \{xb \mid x \in \Pi \text{ and } b \in \Sigma\}$. Σ and B are subrings of Π.

We remark that B and Σ are Specker groups. Note that while Π is pure and is cyclic generated over Π by h_N, Σ_Π is easily seen to be Π-pure in Π but not to be cyclic. In general one cannot say any more about a cyclic module than that it is generated by a single element. It turns out that for Π-modules however, the cyclic pure Π-submodules of Π are just the direct summands of Π. This characterization of the direct summands of Π is of the type of theorem poved by O'Neil in [6] for products of groups. This leads us to study the next natural question: When is a submodule of Π a direct sum of direct summands of Π?

§ 2. The main results

In what follows let ω denote an arbitrary subring of Π which contains B.

We begin by stating the following two easy but basic theorems:

THEOREM A. *The rings Π and B have the property that any finitely generated Π (resp. B)-submodule of Π (resp. B) is cyclic.*

We remark that if $|N|$ is infinite then there are uncountably many distinct rings ω which lie between Π and B. Therefore such a generalization from Π or B modules to ω-modules seems justified.

We shall make extensive use of the next theorem which gives a characterization of ω-purity which has far reaching consequences and which is not available for Z-modules. Although the theorem shows that the notions of purity and primness are equaivalent for ω-modules, it will frequently be important to view purity from the aspect of primness. For this reason we shall retain both notions.

THEOREM B. *Concerning an ω-submodule M of ω we have the following:*
(a) *M is ω-pure if and only if M is prim.*

(b) *M is ω-pure if and only if M is generated by characteristic elements.*

Note that Theorem B implies that if ω' is a subring of Π and if M is also an ω'-submodule of ω', then M is ω-pure if and only if m is ω'-pure.

The next result characterizes direct summands of ω. This characterization is analogous to that given by O'Neill for direct products of groups in [6].

THEOREM C. *The following three conditions for an ω-submodule M of the ω-module ω are equivalent*:

(a) *M is an ω-direct summand of ω.*

(b) *M is cyclic over ω generated by a characteristic element.*

(c) *M is ω-pure and is finitely generated (as an ω-module).*

We have already noted that Σ_Π is not Π-cyclic. Thus although Σ_Π is clearly a direct sum of cyclic direct summands of Π, it follows from Theorem C that Σ_Π itself is not a direct summand of Π.

By Theorem C, a direct summand of Π which is a direct sum of direct summands of Π has the form $\bigoplus_{d\in D} \Pi h_{X_d}$ where each X_d is a subset of N and $X_d \cap X_e$ is empty if $d \neq e$.

Note that Σ_Π is a direct sum of direct summands of Π which is not contained as a direct summand in a larger direct sum of direct summands of Π. In general a direct sum $A = \bigoplus_{d\in D} \Pi h_{X_d}$ is not contained as a direct summand in a larger direct sum B_1 of direct summands of Π, i.e. $B_1 = A \oplus C$, where C is a direct summand of Π if and only if $\bigcup_{d\in D} X_d = N$.

In the corollary to the next theorem we give a complete description of the ω-submodules of ω which are direct sums of direct summands of ω in our main case in which the cardinality $|N|$ of N is at most countable.

THEOREM D. *Let M denote an ω-submodule of ω. Then we have the following*:

(a) *M is a direct sum of direct summands of ω if and only if M is a direct sum of pure cyclic submodules of ω.*

(b) *If M is a direct sum of direct summands of ω then M is ω-pure and is generated by at most $|N|$ elements.*

(c) *If M is ω-pure and is generated by at most countably many elements, then M is a direct sum of direct summands of ω.*

COROLLARY E. *If N is at most countable then M is a direct sum of direct summands of ω if and only if M is ω-pure and is generated by at most $|N|$ elements.*

Remark. If $|N|$ is countable there may be ω-pure submodules T of ω which are not countably generated. Let $P(N)$ be the set of subsets of N and let S denote a subset of $P(N)$ which is closed under the operations of taking subsets of elements of S and of forming finite unions of elements of S. The existence of T is equivalent to the existence of an S having no countable subset C with the property that every element of S is contained in an element of C (when the elements are viewed as subsets of N). The collection S is an ideal in the Boolean algebra $P(N)$. If C existed it would generate S. It is well known that there are ideals of $P(N)$ which are not countably generated.

For our main case in which $N=Z$ we have the following satisfying characterization of ω-purity. It is analogous to Kulikov's theorem describing any abelian group as an increasing union of direct sums of cyclic groups.

THEOREM F. *Let N denote the set of integers, and assume the continuum hypothesis. Then any ω-submodule of ω is ω-pure if and only if it is a union of an increasing chain of ω-pure submodules of ω each of which is an at most countable direct sum of pure cyclic direct summands of ω.*

The next theorem has no analogue in the theory of Z-modules. Although it is just a restatement of Lemma 8, it deserves to be stated as a theorem.

THEOREM G.

(a) *The interesection of an arbitrary family of ω-pure ω-submodules of ω is ω-pure.*

(b) *The ω-submodule of ω which is generated by an arbitrary family of ω-pure ω-submodules of ω is ω-pure.*

Notation. It follows from Theorem C that for any given subset S of ω there is a unique smallest ω-pure ω-submodule of ω which contains S. This module will be denoted by $P(S)$ or by $P_\omega(S)$ when it is important to refer to ω.

The next theorem is of the type of theorem proved by Bergman, see [1].

THEOREM H.

(a) *Any ω-pure module is the* (*module*) *direct product of its primary components and a torsion free one.*

(b) *Each component in* (a) *is generated* (*over ω*) *by a free subgroup of B having characteristic basis elements.*

(c) *In view of* (a) *and* (b) *any ω-pure submodule of ω is generated by a "free" subgroup of B having characteristic basis elements.*

Notation. If S is a subset of ω let $\langle s\rangle$ (or $\langle s\rangle_\omega$) denote the ω-module generated by S, and let ωS be the set $\{ws \mid w\in\omega,\ s\in S\}$.

Before stating the next theorem note that the map η from the set of b-submodules of B into the set of Π-submodules of Π given by $\eta(E)=\langle E\rangle_\Pi$ is in general neither one-one nor onto. To show that it is not one-one, let $N=\{1\}$, and let $R_1=Q$ the ring of rational numbers. Then $B=Z$. Since $\eta(Z)=\eta(2Z)$, η is not one-one. To show that η is not necessarily onto, let Π be the standard one in which $N=Z=R_i$ for each $i\in N$.

The Π-module generaled by $x=(1, 2, 3, \cdots)$ namely Πx, is easily seen not to be in the image of η.

In view of the above, we now state the following surprising theorem. As will be seen later, it is a powerful and useful reduction theorem. In the main case in which each R_i has zero characteristic, the theorem in effect reduces the study of the pure Π-submodules of Π (or in general ω-submodules of ω) to the study of the pure B-submodules of B. Since each R_i has zero characteristics, this B is just the standard B which is the set of bounded sequences of integers.

Now fix ω and let α associate with each B-pure submodule K of B the ω-module $\alpha(K)$ given by $\langle K\rangle_\omega$. It will follow from the next theorem that $\alpha(K)$ is ω-pure, and hence in this case that $\alpha(K)=P_\omega(K)$.

THEOREM I. *The map α given by $\alpha(K)=\langle K\rangle_\omega$ is a map from the set of B-pure B-submodules of B into the set of ω-pure ω-submodules of ω. The map α has the following properties*:

(1) *α is one to one and onto.*

(2) *$\alpha(K)=\omega K$.*

(3) *The inverse β of α is given by $\beta(L)=L\cap B$.*

(4) *α is completely lattice preserving in the sense that*

(4a) *$\alpha(\bigcap_{c\in C} K_c)=\bigcap_{c\in C}\alpha(K_c)$ where C is an (arbitrary) indexing set and K_c is a B-pure B-submodule of B, and*

(4b) *$\alpha(\langle\bigcup_{c\in C} K_c\rangle_B)=\langle\bigcup_{c\in C}\alpha(K_c)\rangle_\omega$ the notation being as in* (4a).

§ 3. The proofs

Proof of Theorem A. We first prove the theorem for Π. Let $a=\{a_i\}_{i\in N}$ and $b=\{b_i\}_{i\in N}$ be elements of Π and let M be the ideal in Π which they generate. Since for each $i\in N$, R_i is euclidean, there are elements c_i and d_i in R_i such that, ignoring units, e_i the greatest common divisor of a_i and b_i has the form $e_i=(a_i, b_i)=c_ia_i+d_ib_i$. Let $e=\{e_i\}_{i\in N}$, $c=\{c_i\}_{i\in N}$, $d=\{d_i\}_{i\in N}$. Since M is an ideal, it follows that the element $e=ca+db$ is in M. Since a and b are multiples of e, it follows that the ideal generated by a and b is the same as the one generated by e. The case of finitely many generators follows from this. The statement regarding B is proved similarly. To do this it suffices to recall that Z and Z_p are euclidean rings, and that the elements of B are functions whose values lie in rings isomorphic to Z or Z_p where p is prime.

LEMMA 1. *Let M be an ω-submodule of ω. Let M contain the characteristic elements h_X and h_Y where $X\cup Y\subset N$. Then M contains $h_{X\cup Y}$.*

Proof. Let $X-Y$ denote the set of elements of X which are not in Y. Then $h_{X\cup Y}=h_X+(h_{Y-X})h_Y$ which is in M.

Proof of Theorem B.

(a) Clearly primness implies purity. So let $a=\{a_i\}_{i\in N}$ belong to the ω-pure ω-submodule M of ω, and let $X=\operatorname{supp} a$. Since $\omega\supseteq B$, the equation $ay=a$ has the solution $y=h_X$ in ω. Since M is ω-pure, a solution $y=\{y_i\}_{i\in N}$ must exist with $y\in M$.

For such an element y we must have $a_i y_i = a_i$ for each $i \in X$. Since each R_i is euclidean, $y_i = 1_i$ whenever $a_i \neq 0$. Since M is an ω-module, and $h_X \in B \subseteq \omega$, $h_X y$ is in M. But $h_X y = h_X$. This proves that M is prim.

(b) If M is ω-pure then M is prim and hence is clearly ω-generated by characteristic elements. So suppose that M is generated by characteristic elements. It suffices by (a) to show that M is prim.

To this end, let $x_1, x_2, \cdots, x_n$ be finitely many elements of M. Since M is closed under multiplication by elements of B, and hence is closed under multiplication by characteristic functions, it suffices to show that M contains the element $h_{(\operatorname{supp} x_1) \cup (\operatorname{supp} x_2) \cup \cdots \cup (\operatorname{supp} x_n)}$. But his is a consequence of Lemma 1.

LEMMA 2. *Let M be an ω-pure submodule of ω which contains the finitely many elements $x_1, x_2, \cdots, x_n$, and let $X = \operatorname{supp} x_1 \cup \cdots \cup \operatorname{supp} x_n$. Then M contains h_X. Further if M is ω-generated by $x_1, \cdots, x_n$, then M is ω-generated by h_X.*

Proof. It suffices to prove the first part. By Theorem B part a, M contains $h_{X_1}, \cdots, h_{X_n}$. By Lemma 1, M contains h_X.

The following lemmas will be needed in the proof of Theorem C. Their proofs are included for the readers' convenience. For $x \in \Pi$, x/x means $h_{\operatorname{supp} x}$.

LEMMA 3. *Let $\omega = M_1 \oplus M_2$ as ω-modules, since $\omega \supseteq B$, $I_N \in \omega$. Let $I_N = (I_1, I_2) = I_1 + I_2$, $I_i \in M_i$ be the unique representation of I_N as a sum of an element I_1 of M_1 and an element I_2 of M_2. Then $M_i = \omega I_i = \{xa \mid x \in \omega, a \in I_i\}$, $i = 1, 2$. Thus each M_i is cyclic generated by I_i.*

Proof. Let $x \in \omega$. Then $x = xI_N = xI_1 + xI_2 = (xI_1, xI_2) \in M_1 \oplus M_2$, where $xI_i \in M_i$, since M_i is a submodule, $i = 1, 2$.

LEMMA 4. *The elements I_1 and I_2 of Lemma 3 are characteristic elements, and their supports form a partition of N.*

Proof. Since each M_i is an ω-submodule, $I_1 I_2 \in M_1 \cap M_2 = 0$ and hence $\operatorname{supp} I_1$ and $\operatorname{supp} I_2$ form a partition of N. Thus $I_1 + I_2 = I_N = (I_N)^2 = I_1^2 + I_2^2$. Comparing coordinates one obtains $I_1 = I_1^2$ and $I_2 = I_2^2$. Since in a euclidean domain R_i we have that $x_i = x_i^2$ if and only if $x_i = 0_i$ or $x_i = 1_i$ it follows that I_1 and I_2 are characteristic.

LEMMA 5. *Any finitely generated ω-pure submodule M of ω is generated by a characteristic element, and hence is a module direct summand of ω.*

Proof. By Lemma 2, M is generated over ω by some h_X, $X \subseteq N$. The direct summand complementary to M is clearly ωh_{N-X}, so that $\omega = M \oplus \omega h_{N-X}$.

Proof of Theorem C.

(a) $\Rightarrow$ (b) follows from Lemmas 3 and 4

(b) $\Rightarrow$ (c) is clear

(c) $\Rightarrow$ (a) follows from Lemma 5.

Remark. (a) Projectivity is not equivalent to freeness for Π-modules. To see this, note that if $x \in N$, then $\Pi h_{\{x\}}$ is a direct summand of $\Pi = \Pi_{i \in N} R_i$ which in general is not free.

(b) Σ_Π is Π-pure but is not a direct summand of Π.

LEMMA 6. *If M is ω-pure in ω then $M \cap B$ is B-pure in B, and $\langle M \cap B \rangle_\omega = M$.*

Proof. Let M be ω-pure and let $x \in M \cap B$. By primness (see Theorem B), $x/x \in M$. Clearly $x/x \in M \cap B$. Hence $M \cap B$ is prim. By Theorem B, the B-module $M \cap B$ is B-pure. The last statement follows from the fact that M is generated (over ω) by characteristic elements and such elements are in B.

Remark. The converse of Lemma 6 is false. Consider the standard Π in which $R_i = N = Z$ for each $i \in N$. The Π-submodule M of the standard Π generated by $(1, 2, 3, 4, \cdots)$ and by Σ is not Π-pure, yet $M \cap B$ is just Σ which is B-pure in B.

Notation. If $\{M_t\}_{t \in T}$ is a family of ω-modules, let $\langle \{M_t\}_{t \in T} \rangle$ or $\langle \{M\}_{t \in T} \rangle_\omega$ denote the ω-module generated by all the modules M_t. Thus $\langle \{M_t\}_{t \in T} \rangle = \langle \bigcup_{t \in T} M_t \rangle$, and $\langle M_1, M_2 \rangle$ means $\langle M_1 \bigcup m_2 \rangle$.

LEMMA 7. *Let M_1 and M_2 be pure submodules of the ω-modules ω. Then $\langle M_1, M_2 \rangle_\omega$ and $M_1 \cap M_2$ are ω-pure ω-submodules of ω.*

Proof. Let $x \in M_1 \cap M_2$. Since by Theorem B each M_i is prim, x/x is in each M_i, $i = 1, 2$. Hence $x/x \in M_1 \cap M_2$. Hence $M_1 \cap M_2$ is prim.

Next, let $x \in \langle M_1, M_2 \rangle$. Then $x = y_1 + y_2$, $y_i \in M_i$, $i = 1, 2$. Since M_1 is prim, $y'_1 = h_{(\operatorname{supp} y_1) - (\operatorname{supp} u_2)} \in M_1$ and $y'_2 = h_{\operatorname{supp} y_2} \in M_2$. Hence $x/x = y'_1 + y'_2$ is in $\langle M_1, M_2 \rangle$. Hence $\langle M_1, M_2 \rangle$ is prim.

LEMMA 8. *Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of ω-pure submodules of ω. Then $\bigcap_{\lambda \in \Lambda} M_\lambda$ and $\langle \{M_\lambda\}_{\lambda \in \Lambda} \rangle$ are ω-pure in ω. In particular, given any subset S of ω, there is a unique smallest pure ω-submodule $P(S)$ of ω which contains S.*

Proof. The statement concerning the union follows from Lemma 7, since every element in the union is a finite sum of elements from the M_λ's. The statement concerning the intersection is an easy consequence of primness. Given $x \in \bigcap M_\lambda$, x/x is in each M_λ by Theorem B. Hence $x/x \in \bigcap_{\lambda \in \Lambda} M_\lambda$. Thus $\bigcap_{\lambda \in \Lambda} M_\lambda$ is prim.

LEMMA 9. *Let K be a B-pure B-submodule of B. Then we have the following*: (a) $\langle K \rangle_\omega = \omega K = \{ax \mid a \in \omega, x \in K\}$, (b) $\langle K \rangle_\omega$ *is ω-pure, and* (c) $\langle K \rangle_\omega \cap B = K$.

Proof. (a) Let $y \in \langle K \rangle_\omega$. Then $y = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$, $a_i \in \omega$, $x_i \in K$. Hence $\operatorname{supp} y \subset \operatorname{supp} x_1 \cup \cdots \cup \operatorname{supp} x_n = X$. Since K is prim it contains each characteristic element $h_{\operatorname{supp} x_i}$. By Lemma 2, K contains h_X. Since K is a B-submodule and $X \supseteq \operatorname{supp} y$, $h_{\operatorname{supp} y} \in K$. Since y is a multiple of $h_{\operatorname{supp} y}$, y is of the form ax where $a = y$ and $x = h_{\operatorname{supp} y}$, which is in ωK. Clearly $\langle K \rangle_\omega \supset \omega K$. Thus (a) follows:

(b) The primness of $\langle K\rangle_\omega$ is an immediate consequence of (a) and the fact that K is prim. By Theorem B, $\langle K\rangle_\omega$ is ω-pure.

(c) It suffices to show that $\langle K\rangle_\omega \cap B \subset K$. So let $y \in \langle K\rangle_\omega \cap B$. By (a) $y = ax$ with $a \in \omega$ and $x \in K$. Clearly $\operatorname{supp} y \subset \operatorname{supp} x$. Since K is a B-module which is B-pure, and since $x \in K$, it follows that the characteristic element $h_{\operatorname{supp} y}$ is in K. Hence $yh_{\operatorname{supp} y}$ is in K, since K is a B-module. This says $y \in K$ which completes the proof of the lemma.

LEMMA 10. *For each c in some indexing set C let K_c be a B-pure submodule of B.*

$$\left\{\left\langle \bigcup_{c\in C} (\omega K_c)\right\rangle_\omega\right\} \cap B = \left\langle \bigcup_{c\in C} K_c\right\rangle_B .$$

Then

Proof.
$$\left\langle \bigcup_{c\in C} (\omega K_c)\right\rangle_\omega \cap B \supseteq \left\langle \bigcup_{c\in C} [(\omega K_c)\cap B]\right\rangle_B = \left\langle \bigcup_{c\in C} K_c\right\rangle_B ,$$

where the last step follows from Lemma 9. Now let $x \in \langle \bigcup_{c\in C} (\omega K_c)\rangle_\omega \cap B$. Then $x \in \langle \bigcup_{c\in C} (\omega K_c)\rangle_\omega$. Since each ωK_c is prim, clearly $h_{\operatorname{supp} x} \in \langle \bigcup_{c\in C} K_c\rangle$. But $h_{\operatorname{supp} x} \in B$. Since $x \in B$, it follows that $x \in \langle \bigcup_{c\in C} K_c\rangle_B$.

Proof of Theorem I. The fact that α maps B-pure into ω-pure follows from part (b) of Lemma 9. The fact that α is 1-1 follows from part (c) of Lemma 9. That $\alpha(K) = \omega K$ is just part (a) of Lemma 9. The fact that α is onto follows from Lemma 6. Part (3) of Theorem I follows from parts (1) and (2) of Theorem I and part (c) of Lemma 9 or Lemma 6.

Before proving part (4) of Theorem I note that by Lemma 8 the expressions involved are ω-pure or B-pure as indicated.

To see (4a) note that by Lemmas 8 and 9 each of $\alpha(\bigcap K_c)$ and $\bigcap \alpha(K_c)$ is ω-pure, and since β is one-one, it suffices by part 3 of Theorem I to show that $(\alpha(\bigcap_{c\in C} K_c)) \cap B = (\bigcap \alpha(K_c)) \cap B$. By Lemma 9, $(\bigcap \alpha(K_c)) \cap B = \bigcap(\alpha(K_c) \cap B) = \bigcap(K_c)$. Since by Lemma 8, $\bigcap(K_c)$ is B-pure it again follows from part (2) of Theorem I and Lemma 9 that $(\alpha(\bigcap(K_c))) \cap B = \bigcap(K_c)$. This proves (4a).

To see (4b) note that by Lemma 8, $\langle \bigcup K_c\rangle$ is B-pure. Hence $(\alpha\langle \bigcup_{c\in C} K_c\rangle) \cap B = \langle \bigcup_{c\in C} K_c\rangle$. Similarly $\langle \bigcup_{c\in C} \alpha(K_c)\rangle \cap B = \langle \bigcup_{c\in C} K_c\rangle$ by Lemma 10.

Proof of Theorem H.

Proof of (a): (a) is a consequence of the fact that ω contains B.

Proof of (c): (c) is an obvious combination of (a) and (b).

Proof of (b): In order to prove (b) we may without loss of generalities assume that the euclidean rings R_i all have a common characteristic.

If the common characteristic of the rings R_i is a prime p, then the additive group of M is a vector space over $Z(p)$. Multiplying each basis element $\{a_i\}_{i\in N}$ by $\{a_i'\}_{i\in N}$ where $a_i'=a_i^{-1}$ if $a_i\neq 0$ and $a_i'=0$ otherwise changes the basis elements into the required characteristic functions.

If the common characteristic is zero, then B is just a Specker group, see our introductory remarks in § 1 or see section 97 of [2]. By Theorem I, the ω-pure module M is generated (over ω) by the B-pure B-module $M\cap B$. By Theorem B, $M\cap B$ is generated by characteristic functions. Hence $M\cap B$ is a Specker group. By [2] § 97, $M\cap B$ is free with a characteristic basis.

The proof of Theorem D.

(a) follows from Theorem C.

(b) Let $M=\bigoplus_{d\in D} M_d$ as in (b), where D is some indexing set and each M_d is an ω-direct summand of M. By part (b) of Theorem C, M_d is generated by a characteristic function f_d having support X_d. Since the above decomposition is direct, and since $\omega\supset B$, it is clear that $X_d\cap X_e$ is empty if d and e are distinct elements of D. Since the supports $\{X_d\}_{d\in D}$ are pairwise disjoint, their number cannot exceed $|N|$. Thus the set of functions f_d has cardinal $\leq |N|$ and generates M. The purity of M follows since each element of M is contained in finitely many M_d's and hence is contained in a direct summand of ω which is also a direct summand of M.

(c) Suppose that M is ω-pure and that it is generated by the countably many elements $x_1, x_2, x_3, \cdots$. By Theorem B we may assume that each x_i is characteristic. We shall now use induction to construct a new set of generators $y_1, y_2, y_3, \cdots$ of M having the following properties:

(1) each y_i is characteristic

(2) $\operatorname{supp} y_i \subset \operatorname{supp} y_{i+1}$

(3) For each $m\in Z$, $\langle x_1, x_2, \cdots, x_m\rangle_\omega \subseteq \langle y_1, y_2, \cdots, y_m\rangle_\omega = \langle y_m\rangle_\omega$.

Let $y_1=x_1$, and suppose that $y_1, \cdots, y_n$ have been defined to satisfy (1), (2) and (3).

By Lemma 2, M contains $h_{(\operatorname{supp} y_n \cup \operatorname{supp} x_{m+1})}$. Certainly the element $y_{n+1}=h_{(\operatorname{supp} y_n \cup \operatorname{supp} x_{n+1})}$ satisfies our induction hypothesis.

Next we inductively construct pairwise disjoint sets $X_1, X_2, X_3, \cdots$ as follows: Let $X_1=\operatorname{supp} y_1$, and for $i>1$ let $X_i=(\operatorname{supp} y_i)-(\operatorname{supp} y_{i-1})$. The sets $X_1, X_2, X_3, \cdots$ are clearly pairwise disjoint, and hence the module generated by the characteristic elements h_{X_i}, $i\in Z$ is the direct sum of the pure cyclic modules $M_i=\omega h_{X_i}$, $i\in Z$. Since for each $n\in Z$ we have $\operatorname{supp} y_n=\operatorname{supp} h_{X_1}\cup\cdots\cup\operatorname{supp} h_{X_n}$, it follows that $\langle y_1, \cdots, y_n\rangle=\langle h_{X_1}, \cdots, h_{X_n}\rangle=M_1\oplus\cdots\oplus M_n$. Hence $M=\bigoplus_{n\in Z} M_n$ which completes the proof of Theorem D.

Proof of Corollary E. This corollary is an immediate consequence of (b) and (c) of Theorem D.

Proof of Theorem F. One half of the theorem follows from the fact that the union of a non-decreasing chain of pure submodules is pure.

To prove the converse we assume that M is ω-pure, and prove the theorem first in the case in which $\omega = B$. Since N is countable by our assumption of the continuum hypothesis $|B| \leq \aleph_1$ and the set S of characteristic elements of M can be initially ordered, i.e. well ordered with the additional property that for each b in S the segment $S'_b = \{x \in S \mid x \leq b\}$ is at most countable.

By Theorem C the B-submodule S_b generated by S'_b is B-pure, and since M is B-pure clearly $M = \bigcup_{b \in S} S_b$, and $S_b \subseteq S_c$ if $b \leq c$. By part (c) of Theorem D each S_b is a direct sum of direct summands of ω. Direct summands are cyclic by Theorem C. Thus M is the non-decreasing union of the chain of B-pure B-modules S_b. In what follows it is helpful to keep in mind that S_b has the form $\bigoplus_{c \in C} E_c$ where C consists of mutually disjoint subsets of N, and $E_c = Bh_c$ where h_c is the characteristic function having support c, see the proof of (c) of Theorem D.

We now take up the general case in which ω is not necessarily B. By Theorem I we have that $M = \omega(M \cap B)$ and that $M \cap B$ is B-pure. By the special case already proved $M \cap B$ is a union of a non-decreasing chain of B-modules S_b each of which is an at most countable direct sum of direct summands S^j_b, $j \in J_b$ of B, where J_b is an indexing set. Let h^j_b be the characteristic function generating S^i_b. The existence of S^i_b is guaranteed by Theorem C. Clearly $\operatorname{supp} h^i_b$ is disjoint from $\operatorname{supp} h^j_b$ if $i \neq j$, and $S_b = \bigoplus_{j \in J_b} Bh^j_b$ where $Bh^j_b = \{xh^j_n \mid x \in B\}$. Since $M = \omega(M \cap B)$ it is equally clear that M is the union of the modules ωS_b, and that since $\omega \supset B$ each ωS_b has the form $\bigoplus_{j \in J_b} (\omega h^j_b)$ which is the form sought in the theorem.

§4. Various properties of ω-pure submodules of ω

(1) If M is an ω-submodule of ω then the purification $P(M)$ of M, see §1, is given in §1 by $P(M) = \{a(x/x) \mid a \in \omega, x \in M\}$. This is an immediate consequence of Theorem B.

(2) If M and V are disjoint submodules of ω then $P(M) \cap P(V) = 0$. This follows from (1) since $M \cap V = 0$ implies that the set of supports of elements of M is disjoint from the set of supports of elements of V.

(3) If M is cyclic it follows from (1) that $P(M)$ is cyclic. But the converse of this is false: Let Π be the standard one in which $N = Z$ and each $R_i = Z$. Let M be the Π-submodule of Π generated by Σ and $2\Pi = \{2x \mid x \in \Pi\}$, where if $x = \{x_i\}_{i \in N}$ then $2x = \{2x_i\}_{i \in N}$. It is not hard to show M is not cyclic and that $P(M) = \Pi$ which is cyclic.

(4) The only indecomposable ω-submodules of ω are those modules each element of which has a support consisting of a single element.

(5) Π is cyclic over Π and it is not $\aleph_0$-decomposable. Yet Σ is a submodule of it which is not cyclic but which is $\aleph_0$-decomposable.

(6) The only basic submodule of Σ is Σ.

(7) The following is an example of a Π-submodule of Π which is not a direct

sum of cyclic Π-submodules of Π. Let Π and M be as in (3) of this section. It is not hard to verify that M is not a direct sum of cyclic Π-submodules of Π.

We conclude with the following proglem: Investigate the "Π-basic" submodules of Π.

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